Complex Analysis: Midterm Exam

Aletta Jacobshal 01, Monday 19 December 2016, 09:00–11:00
Exam duration: 2 hours

Instructions — read carefully before starting

- Write very clearly your **full name** and **student number** at the top of the first page of your exam sheet and on the envelope. **Do NOT seal the envelope!**
- Solutions should be complete and clearly present your reasoning. If you use known results (lemmas, theorems, formulas, etc.) you must explain why the conditions for using such results are satisfied.
- 10 points are "free". There are 5 questions and the maximum number of points is 100. The exam grade is the total number of points divided by 10.
- You are allowed to have a 2-sided A4-sized paper with handwritten notes.

Question 1 (22 points)

Consider the function

$$f(z) = u(x, y) + iv(x, y) = e^x(x\cos y - y\sin y) + ie^x(y\cos y + x\sin y).$$

(a) (12 points) Prove that $f(z)$ is entire using the Cauchy-Riemann equations *Hint: the Cauchy-Riemann equations do not allow by themselves to claim that the function is* analytic*; more conditions must be satisfied and they should be part of your answer.*

Solution

We check the Cauchy-Riemann equations. We have

$$\frac{\partial u}{\partial x} = e^x(x\cos y - y\sin y) + e^x\cos y,$$

and

$$\frac{\partial v}{\partial y} = e^x(\cos y - y\sin y + x\cos y).$$

Moreover,

$$\frac{\partial u}{\partial y} = e^x(-x\sin y - \sin y - y\cos y),$$

and

$$\frac{\partial v}{\partial x} = e^x(y\cos y + x\sin y) + e^x\sin y.$$

Therefore, we see that the Cauchy-Riemann equations

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x},$$

are satisfied.

Moreover, the partial derivatives exist and are continuous for all $x + iy \in \mathbb{C}$. This implies that f is differentiable on $\mathbb{C}$ and since $\mathbb{C}$ is an open set we conclude that f is entire.

(b) (10 points) Write $f(z)$ as a function of z (instead of x and y separately).

Solution

We rearrange terms in $f(z)$ and we find

$$\begin{aligned} f(z) &= e^x \cos y(x+iy) + e^x \sin y(-y+ix) = e^x \cos y(x+iy) + ie^x \sin y(x+iy) \\ &= e^x(\cos y + i\sin y)(x+iy) = e^x e^{iy}(x+iy) = e^{x+iy}(x+iy) \\ &= e^z z. \end{aligned}$$

Question 2 (18 points)

The principal value of arctan is defined as

$$\operatorname{Arctan}(z) = \frac{i}{2}\operatorname{Log}\frac{i+z}{i-z}.$$

(a) (6 points) Compute $\operatorname{Arctan}(-1)$ using the definition of $\operatorname{Arctan}(z)$.

Solution

We have

$$\begin{aligned} \operatorname{Arctan}(-1) &= \frac{i}{2}\operatorname{Log}\frac{i-1}{i+1} \\ &= \frac{i}{2}\operatorname{Log}\frac{(-1+i)(1-i)}{(1+i)(1-i)} \\ &= \frac{i}{2}\operatorname{Log}\frac{2i}{2} \\ &= \frac{i}{2}\operatorname{Log} i \\ &= \frac{i}{2}(\operatorname{Log}|i| + i\operatorname{Arg} i) \\ &= \frac{i}{2}\left(\operatorname{Log} 1 + i\frac{\pi}{2}\right) \\ &= -\frac{\pi}{4}. \end{aligned}$$

(b) (12 points) Show that for $x \in \mathbb{R}$ we have $\operatorname{Arctan}(x) \in (-\pi/2, \pi/2)$. *Hint: show that for $x \in \mathbb{R}$ we have $\left|\frac{i+x}{i-x}\right| = 1$.*

Solution

By definition we have

$$\begin{aligned} \operatorname{Arctan}(x) &= \frac{i}{2}\operatorname{Log}\frac{i+x}{i-x} \\ &= \frac{i}{2}\left(\operatorname{Log}\left|\frac{i+x}{i-x}\right| + i\operatorname{Arg}\frac{i+x}{i-x}\right). \end{aligned}$$

We compute that

$$\begin{aligned} \left|\frac{i+x}{i-x}\right|^2 &= \left(\frac{i+x}{i-x}\right)\overline{\left(\frac{i+x}{i-x}\right)} = \left(\frac{i+x}{i-x}\right)\left(\frac{-i+x}{-i-x}\right) = \left(\frac{i+x}{i-x}\right)\left(\frac{-(i-x)}{-(i+x)}\right) \\ &= \left(\frac{i+x}{i-x}\right)\left(\frac{i-x}{i+x}\right) = 1. \end{aligned}$$

Therefore,

$$\left|\frac{i+x}{i-x}\right| = 1,$$

and

$$\operatorname{Log}\left|\frac{i+x}{i-x}\right| = \operatorname{Log} 1 = 0.$$

This implies

$$\operatorname{Arctan}(x) = -\frac{1}{2}\operatorname{Arg}\frac{i+x}{i-x}.$$

Since for any $z \in \mathbb{C}$ we have $-\pi < \operatorname{Arg} z \leq \pi$ we can conclude that

$$\frac{\pi}{2} \leq \operatorname{Arctan}(x) < \frac{\pi}{2}.$$

The equality with $-\pi/2$ can only be valid if

$$\operatorname{Arg}\frac{i+x}{i-x} = \pi \Leftrightarrow \frac{i+x}{i-x} = s$$

for some negative real number s. Then we find

$$i + x = is - sx \Leftrightarrow (s+1)x = i(s-1) \Leftrightarrow x = i\frac{s-1}{s+1}.$$

The last relation cannot hold for any $s < 0$ since it would imply that x is purely imaginary. This means that $\operatorname{Arctan}(x) \neq \pi/2$ and therefore

$$-\frac{\pi}{2} < \operatorname{Arctan}(x) < \frac{\pi}{2}.$$

Question 3 (18 points)

Show that

$$\left|\int_C \frac{e^z}{\bar{z}+2}dz\right| \leq \pi e^2,$$

where C is the positively oriented circle $|z-1| = 1$.

Solution

On C we have that $0 \leq x \leq 2$ where $x = \operatorname{Re} z$. It is possible to see this by drawing C or by noticing that $x - 1 = \operatorname{Re}(z) - 1 = \operatorname{Re}(z-1)$ and we always have $|\operatorname{Re} w| \leq |w|$, so $|x-1| \leq 1$. Therefore,

$$|e^z| = |e^x e^{iy}| = e^x \leq e^2.$$

Moreover,

$$|\bar{z}+2| = |(\bar{z}-1)+3| \geq ||\bar{z}-1|-3| = |1-3| = 2.$$

Therefore,

$$\left|\frac{e^z}{\bar{z}+2}\right| \le \frac{e^2}{2}.$$

This means

$$\left|\int_C \frac{e^z}{\bar{z}+2}\,dz\right| \le \frac{e^2}{2}\ell(C) = \pi e^2,$$

where, in the last step, we used that the length of the circle C of radius 1 is 2π.

Question 4 (18 points)

A function $f(z)$ is analytic in a domain D. Prove that if the modulus $|f(z)|$ is constant in D then the function $f(z)$ is constant in D.

Solution

We write $f(z) = u(x, y) + iv(x, y)$. If $|f(z)|$ is constant in D then $|f(z)|^2 = u^2 + v^2$ is constant in D.

Let $u^2 + v^2 = c$ throughout D. Then

$$\frac{\partial}{\partial x}(u^2 + v^2) = u\frac{\partial u}{\partial x} + v\frac{\partial v}{\partial x} = 0,$$

and

$$\frac{\partial}{\partial y}(u^2 + v^2) = u\frac{\partial u}{\partial y} + v\frac{\partial v}{\partial y} = 0.$$

Using the Cauchy-Riemann equations we get

$$u\frac{\partial u}{\partial x} - v\frac{\partial u}{\partial y} = 0, \quad u\frac{\partial u}{\partial y} + v\frac{\partial u}{\partial x} = 0.$$

Multiply the first equation by u and the second by v and add them together to get

$$(u^2 + v^2)\frac{\partial u}{\partial x} = 0.$$

Then multiply the first equation by v and the second by u and subtract them to get

$$(u^2 + v^2)\frac{\partial u}{\partial y} = 0.$$

We distinguish two cases. First, if $u^2 + v^2 = 0$ (constant throughout D) then $u = 0$ and $v = 0$ in D which implies $f(z) = 0$ is constant in D.

Otherwise, if $u^2 + v^2 = c \neq 0$ we get

$$\frac{\partial u}{\partial x} = \frac{\partial u}{\partial y} = 0,$$

and from here we conclude that u is constant in D. From the Cauchy-Riemann equations we get

$$\frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y} = 0, \quad \frac{\partial v}{\partial y} = \frac{\partial u}{\partial x} = 0,$$

implying that v is also constant in D and, in conclusion, that f is constant in D.

Question 5 (14 points)

Compute the value of the integral

$$\int_\Gamma \frac{\cos(\pi z)}{(z-1)(z-3)^2}\,dz$$

where Γ is the closed contour shown in Figure 1.

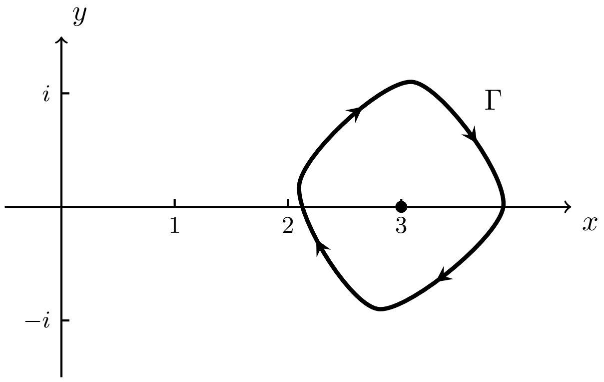

Figure 1: Contour Γ for Question 5.

Solution

The generalized Cauchy integral formula for $n = 1$ and $z_0 = 3$ gives that for a function $g(z)$ analytic on and inside Γ we have

$$\int_\Gamma \frac{g(z)}{(z-3)^2}\,dz = -2\pi i g'(3),$$

where the minus sign comes from the fact that Γ is negatively oriented.

Choosing

$$g(z) = \frac{\cos(\pi z)}{z-1},$$

we note that it is analytic on and inside Γ so it satisfies the conditions for applying the formula.

Therefore

$$\int_\Gamma \frac{\cos(\pi z)}{(z-1)(z-3)^2}\,dz = -2\pi i g'(3).$$

We then compute

$$g'(z) = -\frac{\cos(\pi z) + \pi(z-1)\sin(\pi z)}{(z-1)^2},$$

so

$$g'(3) = -\frac{\cos(3\pi) + 2\pi\sin(3\pi)}{4} = \frac{1}{4}.$$

Finally,

$$\int_\Gamma \frac{\cos(\pi z)}{(z-1)(z-3)^2}\,dz = -2\pi i\,\frac{1}{4} = -\frac{\pi i}{2}.$$